

# Group Theory and the Fifteen Puzzle

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- ▶ Associativity: For all  $a, b, c \in G$ ,  $(a \star b) \star c = a \star (b \star c)$ .

# The Fifteen Puzzle

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5	6	7	8
9	10	11	12
13	14	15	

We move the tiles by sliding the empty slot.

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We move the tiles by sliding the empty slot.

## Question

Which configurations of tiles can we achieve on the Fifteen Puzzle?



# The Fifteen Puzzle (cont.)

## Proposition

The set of moves that leave cell 16 empty on the Fifteen Puzzle forms a group, with the group operation being the composition of moves.

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- ▶ Closure: If  $a, b \in P$ , then  $a * b$  is another scrambled state with cell 16 empty.
- ▶ Identity: The default state is the identity element.
- ▶ Inverse: Every move is reversible.

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Suppose that  $\sigma$  is represented by the following map:

$n$	1	2	3	4	5	6
$\sigma(n)$	4	3	2	6	1	5

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The set of permutations on  $n$  elements forms a group under composition. This group is called the *symmetric group*  $S_n$ .



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## Question

Which properties of permutations relating to their transposition representations are well-defined?



## Definition

A permutation is *even* if it can be written as the product of an even number of transpositions and *odd* if it can be written as the product of an odd number of transpositions.

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## Example

$$A_4 = \{e, (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

# The Fifteen Puzzle Challenge: (14 15)

1	2	3	4
5	6	7	8
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 $\xrightarrow{?}$ 

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## Question

Is it possible to go from the default state to a state with 14 and 15 swapped?

## Proposition

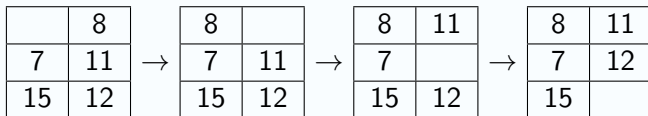
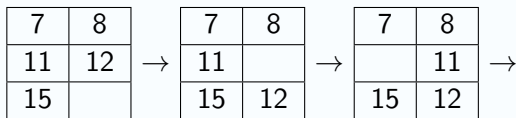
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## Example

This sequence of moves represents the permutation (7 11 8):





$$P < A_{15}$$

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- ▶ Every move is a product of transpositions involving the empty slot:

$$\sigma = \tau_r \tau_{r-1} \cdots \tau_2 \tau_1.$$

- ▶ The number of transpositions  $r$  is even because:
  - ▶ Same number of 'up' and 'down' transpositions
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## Corollary

It is impossible to go from the default state to a state with 14 and 15 swapped.

# Generators of the Alternating Group

## Definition

The set  $\{g_1, g_2, \dots, g_n\}$  *generates* a group  $G$  if all  $g \in G$  can be written as a combination of the  $g_i$  and their inverses.

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For  $n \geq 3$ ,  $A_n$  is generated by the cycles of the form  $(1\ 2\ m)$ , where  $m \in [3, n]$ .



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$A_{15}$  is generated by the 3-cycles  $\{(11\ 12\ 1), \dots, (11\ 12\ 10), (11\ 12\ 13), (11\ 12\ 14), (11\ 12\ 15)\}$ .

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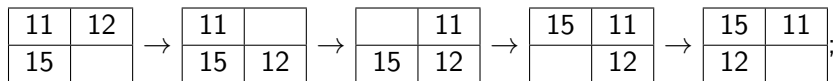
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Proof:



## Lemma

For any permutation  $\rho \in S_{15}$ ,  $\rho^{-1}(i_1 \ i_2 \ i_3)\rho = (\rho^{-1}(i_1) \ \rho^{-1}(i_2) \ \rho^{-1}(i_3))$ .

## Lemma

For any permutation  $\rho \in S_{15}$ ,  $\rho^{-1}(i_1 i_2 i_3)\rho = (\rho^{-1}(i_1) \rho^{-1}(i_2) \rho^{-1}(i_3))$ .

## Proposition

$(11\ 12\ j) \in P$  for  $1 \leq j \leq 15, j \neq 11, 12, 15$ .

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## Proposition

$(11\ 12\ j) \in P$  for  $1 \leq j \leq 15, j \neq 11, 12, 15$ .

By the lemma, if we can find  $\rho_j \in P$  such that

$$\begin{aligned}\rho_j : j &\mapsto 15 \\ 11 &\mapsto 11 \\ 12 &\mapsto 12 \\ 16 &\mapsto 16\end{aligned}$$

then

$$\rho_j^{-1}(11\ 12\ 15)\rho_j = (\rho_j^{-1}(11) \rho_j^{-1}(12) \rho_j^{-1}(15)) = (11\ 12\ j).$$

# $A_{15} < P$ : Constructing $\rho_j$

Consider (11 12 16):

1	2	3	4
5	6	7	8
9	10	16	11
13	14	15	12

Clearly, by design,  $(11\ 12\ 16) \notin P$ . Here are two paths (bold font) the empty slot, 16, can move on so that a new number,  $j$ , would show up at cell 15 while 16 comes back to the same cell:

<b>1</b>	<b>2</b>	<b>3</b>	4
<b>5</b>	6	<b>7</b>	8
<b>9</b>	10	<b>16</b>	11
<b>13</b>	<b>14</b>	<b>15</b>	12

1	<b>2</b>	<b>3</b>	<b>4</b>
5	<b>6</b>	<b>7</b>	<b>8</b>
9	<b>10</b>	<b>16</b>	11
13	<b>14</b>	<b>15</b>	12



Call such a move  $\omega_j$ , which leaves cell 11 empty. As a permutation,  $\omega_j$  fixes cells 11, 12, 16 and send  $j$  to 15. In other words,

$$\omega_j : j \mapsto 15$$

$$11 \mapsto 11$$

$$12 \mapsto 12$$

$$16 \mapsto 16$$

We know the 3-cycle  $(11\ 12\ 16)$  does not affect  $j$  and 15. Thus, if we define  $\rho_j$  as

$$\rho_j = (11\ 12\ 16)^{-1} \omega_j (11\ 12\ 16),$$

then we can see

$$\rho_j : j \mapsto 15$$

$$11 \mapsto 11$$

$$12 \mapsto 12$$

$$16 \mapsto 16$$

and  $\rho_j \in P$  because the empty slot is in cell 16.

Now we know

$$(11\ 12\ j) = \rho_j^{-1}(11\ 12\ 15)\rho_j \in P$$

Thus we have shown

$$\{(11\ 12\ 1), \dots, (11\ 12\ 10), (11\ 12\ 13), (11\ 12\ 14), (11\ 12\ 15)\} \in P,$$

proving

## Theorem

$A_{15}$  is a subgroup of  $P$ .

$$P = A_{15}$$

Since we have proven  $P$  is a subgroup of  $A_{15}$  and  $A_{15}$  is a subgroup of  $P$ , we can conclude:

### Theorem

$$P = A_{15}.$$

# Acknowledgments

We would like to thank the following for their support and guidance throughout this project:

- ▶ Our mentor, Margalit Glasgow
- ▶ Isabel Vogt and the PRIMES Circle program
- ▶ The MIT Math Department
- ▶ Our parents
- ▶ Amtrak and Uber

# Questions?